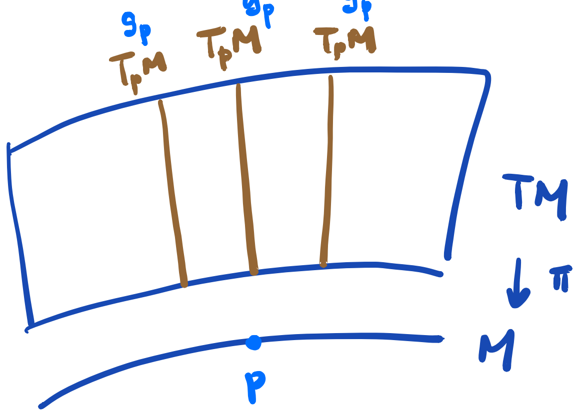
§ Riemannian metrics

Def$^{\underline{n}}$: A Riemannian metric on M^n, denoted by g or $\langle , \rangle$, is an association of an inner product (pos. definite) g_p or $\langle , \rangle_p$ defined on T_pM depending "smoothly" on $p \in M$.

Picture:



Locally, in coord. $x^1, \ldots, x^n$ on M.

$$g = (g_{ij}) \quad \text{symm. pos. definite } n \times n \text{ matrices (of fcn)}$$

where $g_{ij}(x^1, \ldots, x^n) := g_{(x^1,\ldots,x^n)}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$ depends smoothly on $(x^1, \ldots, x^n)$

Equivalently, g is a symmetric $(0,2)$-tensor on M which is pos. definite at every point on M.

Def$^{\underline{n}}$: (M^n, g) Riemannian manifold

Def$^{\underline{n}}$: A smooth map $f : (M, g) \to (N, h)$ between Riem. manifolds is (i) an isometry if $f : M \to N$ is a diffeomorphism s.t. $f^* h = g$ (i.e. $g_p(u,v) = h_{f(p)}(df_p(u), df_p(v))$)

(ii) a local isometry at p if $\exists$ nbd $\mathcal{U}$ of p s.t. $f : \mathcal{U} \to f(\mathcal{U})$ is an isometry

Examples: 1) $\mathbb{R}^n$, $g_{\text{Eucl.}} := \sum_{i=1}^{n} dx^i \otimes dx^i$

2) **Isometric Immersions**: $f: M \to (N, h)$ immersion.

$\Rightarrow$ f^*h is a Riem. metric on M

(Why? $(f^*h)_p(u,v) := h_{f(p)}(\underset{1\text{-}1}{df_p}(u), \underset{1\text{-}1}{df_p}(v))$)

In particular, $M \subset N$, then $\iota: M \hookrightarrow N$ inclusion map.

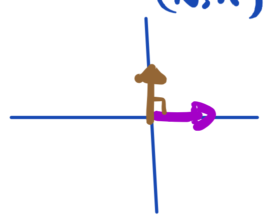
h Riem metric on N $\Rightarrow$ $h|_M := \iota^* h$ induced metric on M.

E.g.) $S^n \subseteq (\mathbb{R}^{n+1}, g_{\text{Eucl.}})$ $\Rightarrow$ $g_{\text{Eucl.}}|_{S^n} =: g_{\text{round}}$.

3) **Product metric**: $(M, g), (N, h)$ Riem mfd

$\rightsquigarrow$ $(M \times N, g \oplus h)$ product Riem. mfd.

locally, $(g \oplus h)_{ij} = \left(\begin{array}{c|c} g_{ij} & 0 \\ \hline 0 & h_{ij} \end{array}\right)$



E.g.) Consider $S^1 \subseteq (\mathbb{R}^2, g_{\text{Eucl.}})$

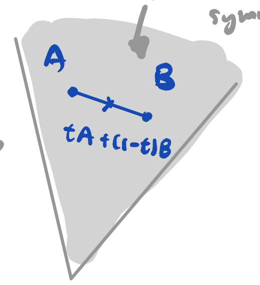
$\rightsquigarrow$ $T^n := \underbrace{S^1 \times \cdots \times S^1}_{n\text{-times}}$ flat n-torus

Prop: Every smooth manifold M^n admits a Riem. metric.

"**Sketch of Proof**": $\{(U_i, \phi_i)\}$ atlas, w. P.O.U. $\{\rho_i\}$.

define: $g := \sum_{i \in I} \rho_i \underbrace{\left(\sum_{j=1}^{n} dx^j \otimes dx^j\right)}_{\text{pos definite}}$

Space of pos-def. symm matrices is a cone



$\square$

Remark: Every oriented (M^n, g) has a preferred volume form

$$\omega = dV_g \overset{\text{loc.}}{=} \underbrace{\sqrt{\det(g_{ij})}\, dx^1 \wedge \dots \wedge dx^n}_{\substack{\text{invar. under} \\ \text{change of coord.}}}$$

Remark: Let $c : [a,b] \to M$ smooth curve.

Define $\text{Length}(c) := \int_a^b \sqrt{g_{c(t)}(c'(t), c'(t))}\, dt$

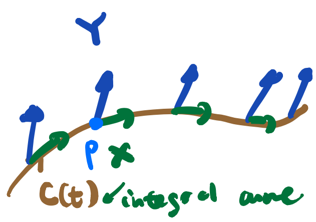
§ Connections

Q: How do we differentiate $f \in C^\infty(M)$?

A: $f \in C^\infty(M) \rightsquigarrow \underbrace{X(f) \text{ or } df(X) \text{ or } \mathcal{L}_X f}_{\text{the same!}}$

Q: How do we differentiate vector fields $Y \in \Gamma(TM)$?

A1: Lie derivative $\mathcal{L}_X Y = [X, Y]$



differentiate Y along c ?

$(\mathcal{L}_X Y)(p)$ depends on the values of X and Y in a nbd. of p

Reason: $\mathcal{L}_X Y$ is NOT tensorial in X.

$$\mathcal{L}_{fX} Y \neq f \mathcal{L}_X Y$$

A2: "Covariant derivative" $\nabla_X Y$

needs an extra structure of a "connection".

Def^{n}: An affine connection on M is a map

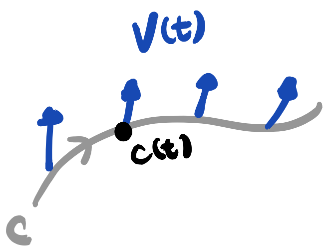
$$\nabla : \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM)$$
$$(X, Y) \longmapsto \nabla_X Y$$

s.t. (i) ∇ is bilinear over $\mathbb{R}$.

(ii) $\nabla_{fX} Y = f \nabla_X Y \quad \forall f \in C^\infty(M)$

(iii) $\nabla_X (fY) = X(f)\, Y + f\, \nabla_X Y \quad \forall f \in C^\infty(M)$

One can use ∇ to define a covariant derivative as follows:



$C(t) : I \to M$ curve

$V(t) : I \to TM$ vector field along c.

i.e. $V(t) \in T_{c(t)}M$

Def^{n}: $\dfrac{DV}{dt} := \underbrace{\nabla_{c'} V}_{\text{a vector field along } c.}$

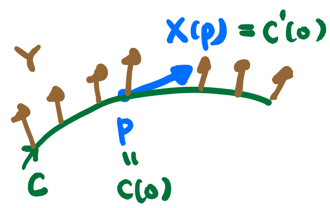
Covariant derivative of V along c

$\because \nabla$ is tensorial in X-variable

$\Rightarrow$ well-definedness.

Lemma: $\nabla_X Y(p)$ depends only on $X(p)$ and the values of Y along ANY curve tangent to X at p.

Proof: Locally in coordinates,



$$X = \sum_i a_i \frac{\partial}{\partial x_i} \quad ; \quad Y = \sum_j b_j \frac{\partial}{\partial x_j}$$

$$\nabla_X Y = \nabla_{\sum_i a_i \frac{\partial}{\partial x^i}}\Big(\sum_j b_j \frac{\partial}{\partial x^j}\Big) \overset{(i)}{\underset{(ii)}{=}} \sum_i a_i \nabla_{\frac{\partial}{\partial x^i}}\Big(\sum_j b_j \frac{\partial}{\partial x^j}\Big)$$

$$\overset{(i)}{=} \sum_{i,j} a_i \nabla_{\frac{\partial}{\partial x^i}}\Big(b_j \frac{\partial}{\partial x^j}\Big) \overset{(iii)}{=} \sum_{i,j} a_i \frac{\partial b_j}{\partial x^i}\frac{\partial}{\partial x^j} + \sum_{i,j} a_i b_j \overset{\text{vector field}}{\boxed{\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}}}$$

Write $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x^k}$, where Γ_{ij}^k = Christoffel symbols of ∇

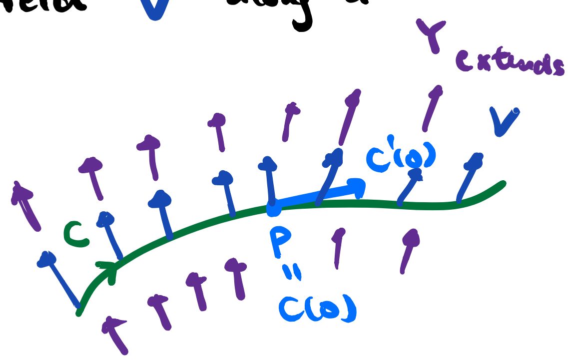
$$\nabla_X Y = \sum_k X(b_k)\frac{\partial}{\partial x^k} + \sum_{i,j,k} \Gamma_{ij}^k a_i b_j \frac{\partial}{\partial x^k}$$

$$= \sum_k \Big[\underbrace{X(b_k)}_{\substack{\text{depends only on}\\ X(p) \text{ and}\\ Y \approx b_k \text{ along a curve } c}} + \underbrace{\sum_{i,j} \Gamma_{ij}^k a_i b_j}_{\substack{\text{depends only}\\ \text{on } X(p),\, Y(p)}}\Big]\frac{\partial}{\partial x^k}$$

at p :

$\square$

This Lemma provides a way to define the notion of "Covariant derivative" of a vector field V along a curve $C(t)$ on M :



$$\underset{\text{at } p}{\frac{DV}{dt}} := \underset{\text{depends only on } c'(0) \text{ and } V.}{\nabla_{c'(t)} Y}$$

where Y is any smooth (local) extension of V

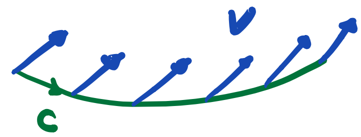
Prop: (1) $\frac{D}{dt}(V+W) = \frac{DV}{dt} + \frac{DW}{dt} \quad \forall\, V, W \in \Gamma(TM).$

(2) $\frac{D}{dt}(fV) = \frac{df}{dt} V + f \frac{DV}{dt} \quad \forall\, f \in C^\infty(M)$

Proof: Follows from (i) – (iii) of ∇.

Def^n: A vector field V along a curve c is parallel

if $\boxed{\frac{DV}{dt} \equiv 0}$



Locally, write $V = \sum_i a_i \frac{\partial}{\partial x^i}$

$c(t) = (c_1(t), \dots, c_n(t))$; $c'(t) = \sum_j c_j'(t) \frac{\partial}{\partial x_j}$

So, V is parallel iff

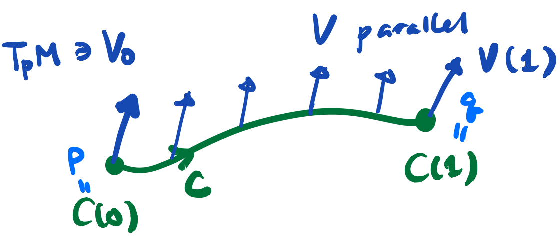
$$0 \equiv \frac{DV}{dt} = \nabla_{c'(t)} V = \nabla_{\sum_j c_j' \frac{\partial}{\partial x^j}} \left(\sum_i a_i \frac{\partial}{\partial x^i} \right)$$

$$\Rightarrow \quad 0 \equiv \sum_k \left(\frac{da_k}{dt} + \sum_{i,j} \Gamma_{ij}^k c_j' a_i \right) \frac{\partial}{\partial x^k}$$

i.e. $$\underbrace{\frac{da_k}{dt}(t) + \sum_{i,j=1}^n \Gamma_{ij}^k (c(t))\, c_j'(t)\, a_i(t) = 0}_{1^{st} \text{ order linear ODE system.}} , \quad \forall\, k = 1, \dots, n$$

$\Rightarrow$ Longtime existence & uniqueness given any initial data $a_i(0)$

In other words, if $c:[0,1]\to M$ is a smooth curve joining $p=c(0)$ to $q=c(1)$. Given any $V_0\in T_pM$, $\exists!$ parallel v.f $V(t)$ along c s.t. $V(0)=V_0$



The map
$$P: T_pM \longrightarrow T_qM$$
$$V_0 \longmapsto V(1)$$
is called parallel transport from p to q along c

Ex: P is a linear isomorphism.

Fact: On the same manifold M, the space of affine connections defined on M forms an affine space, ie

$$\mathrm{Conn.}(M) := \left\{ \begin{array}{c}\text{affine}\\ \text{connections}\\ \text{on } M\end{array} \right\} = T_0 + \underbrace{W}_{\substack{\text{vector}\\ \text{space}}}$$

Motto: For a Riemannian manifold (M,g), there is a "canonical" connection called Riemannian / Levi-Civita connection.

Fundamental Thm. for Riemannian Geometry

Given a Riem. mfd (M^n, g), $\exists!$ connection ∇ s.t. $\forall X, Y, Z \in \mathcal{T}(TM)$

(1) "Metric compatible" ("$\nabla g \equiv 0$")

$$X\big(g(Y,Z)\big) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

(2) "Torsion-free"

$$\nabla_X Y - \nabla_Y X = \overbrace{[X,Y]}^{\text{smooth structure}}$$

Proof: Suppose such a connection ∇ exists.

(1) implies:

$$\begin{aligned}
& X\big(g(Y,Z)\big) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \\
+\ & Y\big(g(Z,X)\big) = g(\nabla_Y Z, X) + g(Z, \nabla_Y X) \\
-\ & Z\big(g(X,Y)\big) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)
\end{aligned}$$

$$\text{R.H.S.} \overset{(2)}{=} g([X,Z],Y) + g([Y,Z],X) + g([X,Y],Z) + 2\, g(\boxed{\nabla_Y X}, Z)$$

Rearranging gives Kozsul formula:

$$g(\nabla_Y X, Z) = \frac{1}{2}\left\{ \begin{array}{l} X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) \\ - g([X,Z],Y) - g([Y,Z],X) - g([X,Y],Z) \end{array} \right\}$$

This shows ∇ exists and is unique! $\square$

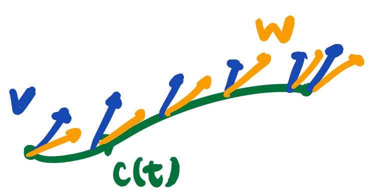
In local coord., $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x^k}$ where

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij} \right)$$

Remarks:

- torsion-free $\Longleftrightarrow$ $\Gamma_{ij}^k = \Gamma_{ji}^k$

  Why? $\underbrace{\nabla_{\partial_i}\partial_j}_{\sum_k \Gamma_{ij}^k \partial_k} - \underbrace{\nabla_{\partial_j}\partial_i}_{\sum_k \Gamma_{ji}^k \partial_k} = [\partial_i, \partial_j] = 0$



- For covariant derivatives along a curve:

$$\frac{d}{dt}\big(g(V,W)\big) = g\left(\frac{DV}{dt}, W\right) + g\left(V, \frac{DW}{dt}\right)$$

In particular, if V, W are parallel, then $g(V, W) \equiv$ const. along c

So, V parallel $\Rightarrow$ $\|V\|^2 := g(V, V) \equiv$ const.

FROM NOW ON, WE ASSUME

$\underset{\text{Riem mfd}}{(M^n, g)}$ equipped with $\underset{\text{Riem. connection}}{\nabla}$